Published for SISSA by Springer

Received: *January 30, 2016*
Revised: *March 10, 2016*
Accepted: *April 21, 2016*
Published: *April 28, 2016*

D1 string dynamics in curved backgrounds with fluxes

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Abstract: We study various rotating and oscillating D-string configurations in some general backgrounds with fluxes. In particular, we look for solutions to the equations of motion of various rigidly rotating D-strings in AdS_3 background with mixed flux, and in the intersecting D-brane geometries. We find out relations among various conserved charges corresponding to the breathing and rotating D-string configurations.

Keywords: AdS-CFT Correspondence, Bosonic Strings, D-branes

ArXiv ePrint: 1601.06360

doi:10.1007/JHEP04(2016)172

JHEP04(2016)172

Contents

1 Introduction and summary

The AdS/CFT correspondence [1–3], relates string states in AdS spaces to some conformal field theory (CFT) living on its boundary. In general, it is very difficult to find the full spectrum of states in the string side and then compare it with the anomalous dimensions of the operators on the CFT side. This has been tough even for the very well studied example of the $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in four dimensions and dual type IIB superstring in the compactified AdS_5 space. One of the direct ways to check the correspondence beyond the supergravity approximation has been to study various classical string solutions in varieties of target space geometries and using the dispersion relation of such strings in the large charge limit, one could look for boundary operators dual to them.[1] This has been one of the main ways to establish the AdS/CFT dictionary in many cases. Instead of probe classical fundamental strings, there have also been many studies on various exact string backgrounds using probe Dp-branes. This provides a novel way of understanding string theory in curved background as the Dp-brane couples to RR fluxes present.

One of the first attempts in this direction was to study D-branes in the $SL(2, \mathbb{R})$ Wess-Zumino-Witten (WZW) model and its discreet orbifolds [5]. The corresponding target space geometries are mainly AdS_3 with non-trivial NS-NS fluxes and 3d AdS black holes [6, 7]. This was extensively studied in a lot of works and the WZW D-branes are now well understood from both CFT boundary states and the target space viewpoint [8–15]. These brane worldvolumes are also shown to carry non-trivial gauge fields. Also coupling to background NS-NS fluxes resists a D-string probe from reducing to a point particle [13, 14].

[1] For a review of semiclassical strings in AdS/CFT one may look at [4].

However, it was also shown that under a supercritical worldvolume electric field, a circular oscillating D-string can never reach the boundary of AdS space [16]. In a related development, D1 strings rotating in $\mathbb{R} \times S^2$ and $\mathbb{R} \times S^3$ were studied in detail by using Dirac-Born-Infeld (DBI) action [17] to find giant magnon [18] and single spike [19] like solutions on the D1-string. Giant magnons arise as rotating solutions in the string side corresponding to low lying spin-chain excitations in the dual field theory. Similarly single spike configurations are particular string solutions which correspond to a particular class of single trace operators in the field theory with large number of derivatives. Both of these appear generally as fundamental string solutions and the D1-string analog in the presence of worldvolume gauge field indeed appears to have yet-unknown novel interpretation.

In general, D1 string solutions in Dp-brane backgrounds are rare due to the complicated nature of the background and presence of dilaton and other RR fields. But this problem appears to be interesting in conjunction to the prediction of [20] that a general Dp-brane background is non-integrable for extended objects. However various well behaved probe brane solutions have been constructed in these backgrounds too. One example is in [21, 22], where the system consisting of two stacks of the fivebranes in type IIB theory that intersect on $\mathbb{R}^{1,1}$ (an Intersecting brane or I-brane [23]) has been investigated using probe D-strings. The much discussed enhancement of symmetry in the near horizon geometry of such systems have been shown to have profound impact on the D1-string worldvolume itself.

Motivated by the above studies, we move on to discuss probe D1 string solutions in a few general settings with coupling to background fluxes. The first study we undertake is a natural generalization of the WZW D-brane solutions. Recently, it was shown that the string theory on $\mathrm{AdS}_3 \times S^3$ supported by both NS-NS and RR three-form fluxes is integrable [24, 25]. There has been proposals of S-matrix on this background [26–32] and various classical string solutions [29, 33–39] have been constructed. The NS-NS flux in this background is parameterized by a number q with $0 \leq q \leq 1$, while the RR 3-form is proportional to $\hat{q} = \sqrt{1-q^2}$. As the q interpolates between 0 to 1, the solution interpolates between that of a pure RR background and a pure NS-NS backgound described by the usual WZW model. However, for intermediate values of q, the exact description of the string sigma model is not known. In any case the study of open string integrability in this background is still lacking, most probably due to dearth of D-brane boundary conditions. However, recently the integrability of a D1-string on the group manifold with mixed three form fluxes has been argued and Lax connections have been constructed [40] with the condition that dilaton and Ramond-Ramond zero forms are constants. It is worth noting that the analysis has only been performed at the bosonic level and it remains to show whether the full action is integrable. Still, even in the bosonic sector, it remains an interesting problem to analyse dynamics of D1-strings in this background. We will probe this backgound with a bound state of oscillating D1 strings and F-strings with non-trivial gauge field on the D1 worldvolume. One of the apparent outcomes of the analysis based on the DBI action is that the periodically expanding and contracting $(1, n)$ string has a possibility of reaching the boundary of AdS_3 in finite time in contrast to the probe D-string motion in the WZW model with only NS-NS fluxes.

JHEP04(2016)172

The later parts of the paper is devoted to study of rotating D1 string solutions in various Dp brane backgrounds. In particular we will talk about the near horizon geometry of a stack of D5 branes and two stacks of D5 branes that intersect on a line. We solve the equations of motion for D1 string keeping couplings with dilatons and WZ terms in mind. It is shown that in proper limits, the combination of properly regularised conserved quantities actually give rise to giant magnon and single spike like dispersion relations as in the case of fundamental strings. We speculate that these exact solutions correspond to S-dual fundamental string solutions in NS5 brane background [41] and in NS5-NS5' brane intersections [42].

The rest of the paper is organised as follows. In section 2 we will discuss the motion of a (m,n) string in the mixed-flux AdS_3 background. In section 3, we will move on to the motion of a rotating D1-string in the near horizon geometry of a stack of D5 branes. We will show that the conserved charges give rise to giant magnon or single spike-like dispersion relations in two different limits. We will discuss a similar configuration in the intersecting D5-D5' brane background in section 4. The equations of motion appear to be very complicated in that case, but with certain simplifications we will be again able to find giant magnon or single spike-like dispersion relations for the properly regularised version of the charges. We conclude with some outlook in section 5.

2 Circular (m,n) strings on AdS_3 with mixed 3-form fluxes

The mixed flux background is a solution of the type IIB action with a $\mathrm{AdS}_3 \times S^3 \times T^4$ geometry, although the compact manifold will not be interesting here. The solution has both RR and NS-NS fluxes along the AdS and S directions. In this section we will put the S^3 coordinates to be constant and consider motion only along the AdS_3. The background and the relevant fluxes are as follows,

$$
\begin{aligned}
&ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\phi^2\,,\\
&B_{(2)} = q\cosh^2\rho\; dt\wedge d\phi\,,\quad C_{(2)} = \sqrt{1-q^2}\cosh^2\rho\; dt\wedge d\phi\,.
\end{aligned} \tag{2.1}
$$

The dilaton Φ is constant and can be set to zero. Also, the AdS radius has been properly chosen here. The above background can easily be shown to be a solution of the type IIB field equations.[2]

Now, we want to discuss the motion of a bound state of m D1-strings and n F-strings in this background as a general string state in type IIB theory would readily be the bound state of the two. For simplicity we will choose $m=1$ and argue that the analysis can be extended to other more general string state too. The DBI action for a D1-brane can be written as,

$$
S = -T_D\int d^2\xi e^{-\phi}\sqrt{-\det(\hat g + \hat B + 2\pi\alpha' F)} + \frac{T_D}{2}\int d^2\xi \epsilon^{\alpha\beta}C_{\alpha\beta}\,. \tag{2.2}
$$

[2]Note that there is a gauge freedom in choosing the two-form B-field since the supergravity equations of motion involves $H_{(3)} = dB_{(2)}$ only. For example, in the case of a three sphere the NS-NS flux is defined upto an additive constant in the form $-\frac{q}{2}(\cos 2\theta + c)$. This constant can be fixed using physical considerations. We have fixed the constant here for the AdS_3 case so that the fields have the mentioned form. For details one could see the arguments presented in [29].

where $\hat{g}$ and $\hat{B}$ are the pullback of the background metric and the NS-NS fluxes and F is the field strength for the worldvolume gauge field. Here T_D is the D1-string tension $T_D = 1/(2\pi\alpha' g_s)$, with g_s being the string coupling. In contrast, remember that the F-string tension was simply $T_F = 1/2\pi\alpha'$. So in weak coupling regime we can write $T_D \gg T_F$. Also, $\epsilon^{\alpha\beta}$ is the usual antisymmetric tensor with $\epsilon^{01} = 1$. Since the mixed flux background has the constant (or zero) dilaton, the Lagrangian density takes the form,

$$\mathcal{L} = -T_D \left[\sqrt{-\det(\hat{g} + \hat{B} + 2\pi\alpha' F)} - \frac{1}{2}\epsilon^{\alpha\beta} C_{\alpha\beta}\right] . \tag{2.3}$$

We choose the following ansatz for a 'breathing' mode of the string:

$$t = \xi^0 , \quad \rho(\xi^0, \xi^1) = \rho(\xi^0) , \quad \phi = \xi^1 . \tag{2.4}$$

The ansatz is like the one for a simple circular F-string motion and can be easliy shown to be consistent with the equations of motion here. Using the above ansatz we can show,

$$\begin{aligned} -\det(\hat{g} + \hat{B} + 2\pi\alpha' F) &= \sinh^2\rho(\cosh^2\rho - (\partial_0\rho)^2) - (q\cosh^2\rho - 2\pi\alpha'\partial_0 A_\phi)^2 \\ &= -\det\,\hat{g} - \mathcal{F}_{\phi t}^2 , \end{aligned} \tag{2.5}$$

where we have

$$\det\,\hat{g} = -\sinh^2\rho(\cosh^2\rho - (\partial_0\rho)^2) \tag{2.6}$$

and

$$\mathcal{F}_{\phi t} = q\cosh^2\rho - 2\pi\alpha'\partial_0 A_\phi . \tag{2.7}$$

Now, we can rewrite the lagrangian density as,

$$\mathcal{L} = -T_D \left[\sqrt{-\det\,\hat{g} - \mathcal{F}_{\phi t}^2} - \sqrt{1 - q^2}\cosh^2\rho\right] . \tag{2.8}$$

Conjugate momentum of the Wilson line A_ϕ is a quantized constant of the motion given by,

$$\frac{1}{2\pi}\Pi_\phi = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_\phi)} = \frac{-2\pi\alpha' T_D \mathcal{F}_{\phi t}}{\sqrt{-\det\,\hat{g} - \mathcal{F}_{\phi t}^2}} = -n . \tag{2.9}$$

The integer n is the number of oriented fundamental strings bound to the D-string. The effective tension of (m, n) circular string is given by,

$$T_{(m,n)} = \sqrt{m^2 T_D^2 + n^2 T_F^2} . \tag{2.10}$$

From the above expression and (2.9) we can write,

$$T_{(1,n)} = T_D \sqrt{1 + \frac{(2\pi\alpha')^2 T_F^2 \mathcal{F}_{\phi t}^2}{(-\det\,\hat{g} - \mathcal{F}_{\phi t}^2)}} . \tag{2.11}$$

Combining all expression we get the following relation which we will later use to simplify other expressions,

$$\frac{T_{(1,n)}}{\sqrt{-\det\,\hat{g}}} = \frac{T_D}{\sqrt{-\det\,\hat{g} - \mathcal{F}_{\phi t}^2}} = \frac{n}{2\pi\alpha' \mathcal{F}_{\phi t}} . \tag{2.12}$$

JHEP04(2016)172

The other constant of the motion is the energy, as measured by an observer sitting at the center of AdS_3,

$$
\begin{aligned}
E &= 2\pi\left(\frac{\partial\mathcal{L}}{\partial(\partial_0\rho)}\partial_0\rho + \frac{\partial\mathcal{L}}{\partial(\partial_0 A_\phi)}\partial_0 A_\phi - \mathcal{L}\right) \\
&= 2\pi T_D\left[\frac{\sinh^2\rho\cosh^2\rho - q\cosh^2\rho(q\cosh^2\rho - 2\pi\alpha'\partial_0 A_\phi)}{\sqrt{-\det\hat{g} - \mathcal{F}_{\phi t}^2}} - \sqrt{1-q^2}\cosh^2\rho\right]. \qquad (2.13)
\end{aligned}
$$

Putting the energy in a more suggestive form we can get,

$$
E = \frac{2\pi T_{(1,n)}\sinh\rho\cosh^2\rho}{\sqrt{\cosh^2\rho - (\partial_0\rho)^2}} - 2\pi\cosh^2\rho\left(nqT_F + \sqrt{1-q^2}T_D\right). \qquad (2.14)
$$

Above equation exhibits the competing terms of the potential energy: the blue-shifted mass, and the interaction with the B-field and C-field potential. Using the above we can write the equation of motion for ρ as,

$$
\partial_0\rho = \frac{\cosh\rho}{E + C_2\cosh^2\rho}\left[(E + C_2\cosh^2\rho)^2 - C_1^2\sinh^2\rho\cosh^2\rho\right]^{1/2} \qquad (2.15)
$$

Where the constants

$$
C_1 = 2\pi T_{(1,n)} \qquad C_2 = 2\pi\left(nqT_F + \sqrt{1-q^2}T_D\right) \qquad (2.16)
$$

Note that $C_1 > C_2$ is still valid as we have $q \in [0, 1]$. Now our interest is to find a large energy solution, i.e. a limit where the string becomes 'long' and tries to reach the boundary of AdS in finite time. In this limit we can solve the radial equation perturbatively in orders of $\frac{1}{E}$. Expanding (2.15), we get,

$$
\partial_0\rho = \cosh\rho - \frac{C_1^2\sinh^2\rho\cosh^3\rho}{2E^2} + \frac{C_1^2 C_2\sinh^2\rho\cosh^5\rho}{E^3} + \mathcal{O}\left(\frac{1}{E^4}\right) \qquad (2.17)
$$

To solve this order by order we take the expression for ρ

$$
\rho = \rho^{(0)} + \frac{\rho^{(1)}}{E^2} + \frac{\rho^{(2)}}{E^3} + \mathcal{O}\left(\frac{1}{E^4}\right) \qquad (2.18)
$$

Putting the above back in (2.15) and expanding we get the following set of coupled differential equations for the different orders

$$
\begin{aligned}
\partial_0\rho^{(0)} &= \cosh\rho^{(0)}, \\
\partial_0\rho^{(1)} &= \rho^{(1)}\sinh\rho^{(0)} - \frac{1}{2}C_1^2\cosh^3\rho^{(0)}\sinh^2\rho^{(0)}, \\
\partial_0\rho^{(2)} &= \rho^{(2)}\sinh\rho^{(0)} + C_1^2 C_2\cosh^5\rho^{(0)}\sinh^2\rho^{(0)}.
\end{aligned} \qquad (2.19)
$$

We solve the first equation to find

$$
\rho^{(0)} = \sinh^{-1}\tan\tau, \qquad (2.20)
$$

which is a periodically expanding and contracting solution where the string goes out to a maximum radius. We use this solution iteratively in the other two equations and find the total solution using the boundary condition $\rho(0) = 0$ to write the perturbative solution

$$\rho(\tau) = \sinh^{-1}\tan\tau - \frac{1}{6E^2}C_1^2\sec\tau\tan^3\tau + \frac{1}{15E^3}C_1^2C_2(4+\cos 2\tau)\tan^3\tau\sec^3\tau + \mathcal{O}\left(\frac{1}{E^4}\right) \tag{2.21}$$

Now the dynamics is incredibly difficult as there are many parameters involved like T_F, T_D, n and q. The dynamics varies widely for different values of these parameters. With higher order corrections (which are suppressed in the large energy limit), the $(1, n)$ string goes through quasi-periodic motion.

Now to find the maximum radius of the circular D-string we note that the extremum would occur at $\partial_0\rho = 0$, which leads us to the following,

$$E + C_2\cosh^2\rho_m - C_1\sinh\rho_m\cosh\rho_m = 0. \tag{2.22}$$

With a little algebra we can find that,

$$\rho_m = \frac{1}{2}\ \ln\left[\frac{2E + C_2 \pm \sqrt{4E^2 + 2EC_2 + C_1^2}}{(C_1 - C_2)}\right]. \tag{2.23}$$

To prove that the above corresponds to the maximum value of ρ we can explicitly show that

$$\partial_0^2\rho(\rho_m) < 0. \tag{2.24}$$

Since we are interested in the large E limit of the solution, we can write the maximum radius approximately in a simpler form by putting in the expressions for $C_{1,2}$,

$$\rho_m \simeq \frac{1}{2}\ \ln\left[\frac{2E}{\pi(T_{(1,n)} - nqT_F - \sqrt{1-q^2}T_D)}\right] \tag{2.25}$$

If one keeps all other parameters fixed and increases q from 0, it can easily be seen that the maximum value of ρ actually increases. So, we can say that with increasing NS-NS flux the large energy string actually gets 'fatter'. However, we are interested to know whether the string becomes a 'long' one. Now interestingly enough, one can see that the expression for maximum radius diverges when

$$T_{(1,n)} \to nqT_F + \sqrt{1-q^2}T_D. \tag{2.26}$$

Note that diverging ρ_m means that the strings can actually expand upto the boundary of the AdS_3. For the case of pure NS-NS flux i.e. $q = 1$ as discussed in [16], the ρ_m can only diverge if $T_{(1,n)} \to nT_F$, so that the string becomes a purely fundamental 'long' string. It was made clear in [16] that a general (m, n) string can never reach the boundary of AdS as the first term in (2.14) diverges faster near the boundary than the NS-NS flux potential term for $q = 1$, making the energy effectively infinite. It is quite clear that for pure NS-NS case the two terms in (2.14) would cancel in the asymptotic region to produce a finite contribution, only if the string is purely fundamental. However here we also have contribution from the

JHEP04(2016)172

RR term, so the string might not need to be purely fundamental to acquire a finite energy near the boundary. In our example, keeping in mind that $T_{(1,n)} = \sqrt{T_D^2 + n^2 T_F^2}$, we can easily see the following,

$$\begin{aligned} & T_D + nT_F > T_{(1,n)}, \\ \text{and} \quad & T_D + nT_F > nqT_F + \sqrt{1-q^2}T_D; \qquad 0 < q < 1. \end{aligned} \tag{2.27}$$

So in any case the condition (2.26) actually could be physical in a particular region of the parameter space and the boundary might not remain forbidden region for the $(1,n)$ string when it couples to both NS-NS and RR fluxes. In fact if we impose an equality in the (2.26), it simply yields the condition

$$n\sqrt{1-q^2}\; T_F = \pm qT_D, \tag{2.28}$$

for the maximum radius of the string to diverge. We shall not be bothering about the negative sign here. It is very clear that in the weak coupling region (small g_s), we can transform the above condition into,

$$n\sqrt{\frac{1-q^2}{q^2}} \gg 1. \tag{2.29}$$

As we have n to be a positive finite integer, the above condition can only be satisfied if the value of q is small. As $q \to 1$, the above inequality is violated since the equality associated with (2.26) loses meaning for this case and can only be true if the string becomes a purely fundamental one. This supports the discussion we presented earlier. This is a unique observation for such strings in this mixed flux background and have to be investigated into details using other approaches.

3 Rotating D1-string on D5-branes

Let us first start with the discussion of generic Dp brane backgrounds. For an arbitrary p, we can write the general supergravity solution as follows,

$$\begin{aligned} ds^2 &= h^{-\frac{1}{2}}(\vec{x})(-dt^2 + d\vec{y}^{\,2}) + h^{\frac{1}{2}}(\vec{x})\; d\vec{x}^{\,2}, \\ e^{2\phi} &= h(\vec{x})^{\frac{3-p}{2}}, \\ C^{p+1} &= \pm\left(\frac{1}{h(\vec{x})} - 1\right)\; dt \wedge dy^1 \wedge \ldots \wedge dy^p, \\ F^{p+2} &= dC^{p+1} = \mp h^{-2}\partial_j h\; dx^j \wedge dt \wedge dy^1 \wedge \ldots \wedge dy^p, \\ \tilde{F}^{8-p} &= \pm \partial_j h\; i_{\hat{x}^j}(dx^1 \wedge \ldots \wedge dx^{9-p}), \\ h(\vec{x}) &= 1 + \frac{\mu}{7-p}\left(\frac{l_s}{r}\right)^{7-p}. \end{aligned} \tag{3.1}$$

Let us remind ourselves the various expressions used in the above equation. Here p spatial coordinates y^k run parallel to the worldvolume while the transverse space is given by $(9-p)$

coordinates labelled by x^i. The specified harmonic function $h(\vec{x})$ is valid for $p = 0, 1, 2 \ldots, 6$ with $r^2 = \sum_{i=1}^{(9-p)} x^i x_i$ being the radial distance along the transverse space. The dilaton is given by ϕ and the RR form is given by C^{p+1}. The RR field strength is given by F^{p+2} while the hodge dual field strength is given by $\tilde{F}^{8-p}$. The action of $i_{\hat{x}^j}$ on the volume form of the transverse space signifies a inner product with a unit vector pointing in the x^j direction.

For a D1-string moving in a general Dp-brane background, the DBI action is given by,

$$S = -T_1 \int d\xi^0 d\xi^1 e^{-\phi} \sqrt{-\det A_{\alpha\beta}} + T_1 \int d\xi^0 d\xi^1 C_{01} \tag{3.2}$$

where $A_{\alpha\beta} = g_{\alpha\beta} = \partial_\alpha X^M \partial_\beta X^N g_{MN}$ is the induced metric on the worldvolume and $\alpha, \beta = \xi^0, \xi^1$. We here put the worldvolume gauge field to be zero for simplicity. C_{01} is the two form RR potential coupled to the D1 string.

The background in which we are interested in is a stack of N D5-branes, which is given by the following metric, dilaton and RR six-form:

$$\begin{aligned} ds^2 &= h^{-\frac{1}{2}} \left[-dt^2 + \sum_{i=1}^{5} dx_i^2 \right] + h^{\frac{1}{2}} \left[dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2) \right] \\ e^\phi &= h^{-\frac{1}{2}}, \quad C_{012345} = \frac{k}{k + r^2}, \quad h = 1 + \frac{k}{r^2}, \quad k = g_s N l_s^2. \end{aligned} \tag{3.3}$$

Here l_s is the string length scale and N is the number of branes in the stack. In the near horizon limit $r \to 0$, so in the harmonic function, the second term dominates and it becomes $h = 1 + \frac{k}{r^2} \approx \frac{k}{r^2}$ and also the six form potential $C_{012345} = 1 + \mathcal{O}(r^2)$. We now rescale the coordinates in the following way

$$t \to \sqrt{k} t \qquad x_i \to \sqrt{k} x_i, \tag{3.4}$$

and following this, the metric and fields reduce to the form,

$$\begin{aligned} ds^2 &= \sqrt{k} r \left[-dt^2 + \sum_{i=1}^{5} dx_i^2 + \frac{dr^2}{r^2} + d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2 \right] \\ e^\phi &= \frac{r}{\sqrt{k}}, \quad C_{012345} = 1, \quad k = g_s N l_s^2. \end{aligned} \tag{3.5}$$

This six form background RR field does not couple to a D1-string. However, the dual of this six form i.e. the 'magnetic' two form RR field ($\tilde{C}_{\phi_1 \phi_2}$) will couple to the of D1-string as WZ contribution.[3] For our convenience we define $\rho = \ln r$, and transform the the metric and the dilaton to,

$$\begin{aligned} ds^2 &= \sqrt{k} e^\rho \left[-dt^2 + \sum_{i=1}^{5} dx_i^2 + d\rho^2 + d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2 \right] \\ e^\phi &= \frac{e^\rho}{\sqrt{k}}, \quad \tilde{C}_{\phi_1\phi_2} = 2k \sin^2\theta, \quad k = g_s N l_s^2. \end{aligned} \tag{3.6}$$

[3]For detailed discussion on this point one could see the discussion in [43].

JHEP04(2016)172

Now, it is to be noted that the RR two form ($\tilde{C}_{\phi_1\phi_2}$) has been written upto an additive constant as we can calculate the field strength $F^{(3)}$ from the (3.1). This 3-form turns out simply to be the volume form of the transverse sphere itself. Now for our probe D1 string, the induced worldvolume metric components are given by,

$$
\begin{aligned}
A_{00} &= \sqrt{k}e^{\rho}\Big[-(\partial_0 t)^2+\sum(\partial_0 x_i)^2+(\partial_0\rho)^2+(\partial_0\theta)^2+\sin^2\theta(\partial_0\phi_1)^2+\cos^2\theta(\partial_0\phi_2)^2\Big]\\
A_{11} &= \sqrt{k}e^{\rho}\Big[-(\partial_1 t)^2+\sum(\partial_1 x_i)^2+(\partial_1\rho)^2+(\partial_1\theta)^2+\sin^2\theta(\partial_1\phi_1)^2+\cos^2\theta(\partial_1\phi_2)^2\Big]\\
A_{01} &= A_{10}=\sqrt{k}e^{\rho}\Big[-(\partial_0 t)(\partial_1 t)+\sum(\partial_0 x_i)(\partial_1 x_i)+(\partial_0\rho)(\partial_1\rho)+(\partial_0\theta)(\partial_1\theta)\\
&\quad+\sin^2\theta(\partial_0\phi_1)(\partial_1\phi_1)+\cos^2\theta(\partial_0\phi_2)(\partial_1\phi_2)\Big]\,. \qquad (3.7)
\end{aligned}
$$

Therefore we can write the total Lagrangian-density as,

$$
\begin{aligned}
\mathcal{L} &= -T_1 e^{-\phi}\sqrt{-\det A_{\alpha\beta}}+\frac{T_1}{2}\epsilon^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N C_{MN}\\
&= -T_1 k\Bigg[\Big[-(\partial_0 t)(\partial_1 t)+\sum(\partial_0 x_i)(\partial_1 x_i)+(\partial_0\rho)(\partial_1\rho)+(\partial_0\theta)(\partial_1\theta)\\
&\quad+\sin^2\theta(\partial_0\phi_1)(\partial_1\phi_1)+\cos^2\theta(\partial_0\phi_2)(\partial_1\phi_2)\Big]^2-\Big[-(\partial_0 t)^2+\sum(\partial_0 x_i)^2\\
&\quad+(\partial_0\rho)^2+(\partial_0\theta)^2+\sin^2\theta(\partial_0\phi_1)^2+\cos^2\theta(\partial_0\phi_2)^2\Big]\Big[-(\partial_1 t)^2\\
&\quad+\sum(\partial_1 x_i)^2+(\partial_1\rho)^2+(\partial_1\theta)^2+\sin^2\theta(\partial_1\phi_1)^2+\cos^2\theta(\partial_1\phi_2)^2\Big]\Bigg]^{\frac{1}{2}}\\
&\quad+2T_1 k\sin^2\theta(\partial_0\phi_1\partial_1\phi_2-\partial_0\phi_2\partial_1\phi_1)\,. \qquad (3.8)
\end{aligned}
$$

Before solving the Euler-Lagrange equations,

$$
\partial_0\left(\frac{\partial\mathcal{L}}{\partial(\partial_0 X)}\right)+\partial_1\left(\frac{\partial\mathcal{L}}{\partial(\partial_1 X)}\right)=\frac{\partial\mathcal{L}}{\partial X}\,, \qquad (3.9)
$$

we choose a rotating ansatz for the probe brane,

$$
\begin{aligned}
&t=\kappa\xi^0, \qquad x_i=\nu_i\xi^0, \qquad\qquad i=1,2,3,4,5, \quad \rho=m\xi^0,\\
&\theta=\theta(\xi^1), \quad \phi_1=\omega_1\xi^0+\xi^1, \quad \phi_2=\omega_2\xi^0+\phi_2(\xi^1)\,. \qquad (3.10)
\end{aligned}
$$

The above ansatz can be thought of as the generalized version of the rigidly rotating string parameterisation described in, for example, [19]. Solving the equations of motion for ϕ_1 and ϕ_2 and eliminating $\mathcal{L}$ from the respective equations we get,

$$
\frac{\partial\phi_2}{\partial\xi^1}=\frac{\sin^2\theta[(c_2-2\omega_2\sin^2\theta)\omega_1\omega_2\cos^2\theta+(c_3+2\omega_1\sin^2\theta)\omega_2^2\cos^2\theta-\alpha^2(c_3+2\omega_1\sin^2\theta)]}{\cos^2\theta[(c_3+2\omega_1\sin^2\theta)\omega_1\omega_2\sin^2\theta+(c_2-2\omega_2\sin^2\theta)\omega_1^2\sin^2\theta-\alpha^2(c_2-2\omega_2\sin^2\theta)]}\,. \qquad (3.11)
$$

Again solving for t, we get the equation for θ,

$$
\begin{aligned}
\left(\frac{\partial\theta}{\partial\xi^1}\right)^2 &= \frac{(c_1^2-\kappa^2)(\omega_1\sin^2\theta+\omega_2\partial_1\phi_2\cos^2\theta)^2}{c_1^2\{-\alpha^2+\omega_1^2\sin^2\theta+\omega_2^2\cos^2\theta\}}\\
&\quad-\{\sin^2\theta+(\partial_1\phi_2)^2\cos^2\theta\}\,. \qquad (3.12)
\end{aligned}
$$

where c_1, c_2 and c_3 are carefully chosen integration constants and $\alpha^2=\kappa^2-m^2-\sum_{i=1}^{5}\nu_i\nu^i$.

JHEP04(2016)172

Certainly, the equation of motion for θ is quite complicated and difficult to solve in general due to the presence of all kinds of constants. In what follows we will discuss two limiting cases from the above set of equations corresponding to single spike and giant magnon solutions. In [19] it was shown that the giant magnon and single spike strings can be thought of two limits of the same system of equations. We will show that this is the same for our case also.

Let us outline the procedure to impose conditions on the equations of motion in a simple schematic way. The main problem lies in the behaviour of $\frac{\partial\phi_2}{\partial\xi^1}$ at $\theta = \frac{\pi}{2}$ as there is a $\cos^2\theta$ sitting in the denominator. To talk about the behaviour of $\frac{\partial\theta}{\partial\xi^1}$ in this limit, we will then define

$$\left(\frac{\partial\phi_2}{\partial\xi^1}\right)^2 \cos^2\theta\bigg|_{\theta=\frac{\pi}{2}} = \mathcal{P}^2. \tag{3.13}$$

Now we want to have particular conditions imposed on $\frac{\partial\theta}{\partial\xi^1}$ too. We can see that

$$\frac{\partial\theta}{\partial\xi^1}\bigg|_{\theta=\frac{\pi}{2}} = \frac{c_1^2\alpha^2 - \kappa^2\omega_1^2 + \mathcal{P}^2 c_1^2(\alpha^2 - \omega_1^2)}{c_1^2(-\alpha^2 + \omega_1^2)}. \tag{3.14}$$

There are two very interesting limits that can be considered here. The first one is to demand that $\frac{\partial\theta}{\partial\xi^1} \to 0$ as $\theta \to \frac{\pi}{2}$, which in turn demands that $c_1\alpha = \kappa\omega_1$. Also it is required in this case that $\mathcal{P}^2 \to 0$ i.e. $\frac{\partial\phi_2}{\partial\xi^1}$ has to be regular, which in turn demands that $c_3 = -2\omega_1$. So, this set of relations between the constants will implement the conditions mentioned above. The other limit may be taken as $\frac{\partial\theta}{\partial\xi^1} \to \infty$ as $\theta \to \frac{\pi}{2}$. It also demands that $\frac{\partial\phi_2}{\partial\xi^1} \to \infty$ in the same limit of θ. Both of these can be realised by the set of relations $\alpha = \omega_1$ and $c_3 = -2\omega_1$. We will discuss these particular two conditions while finding out the constants of motion and the relevant dispersion relations among them. We will see that the equations of motion considerably simplify when we put in the relations between the constants.

3.1 Single spike solution

In this section we will follow the procedure outlined in [19] and consider constants of motions appropriately, demanding the following condition is satisfied,

$$\frac{\partial\theta}{\partial\xi^1} \to 0 \quad \text{as} \quad \theta \to \frac{\pi}{2}. \tag{3.15}$$

As we discussed earlier, this will also require $\frac{\partial\phi_2}{\partial\xi^1}$ to be regular. So combining these, we get the relations between the constants $c_1\alpha = \kappa\omega_1$ and $c_3 = -2\omega_1$. Note that the imposed conditions determine the value of integration constants c_1 and c_3 in terms of other parameters, but it doesn't fix the value of c_2. For the time being we can keep c_2 as it is. Later, we will try to fix a value of c_2 using other considerations. Using these the equations (3.11) and (3.12) transform to,

$$\frac{\partial\phi_2}{\partial\xi^1} = \frac{\omega_1(2\omega_2^2 - 2\alpha^2 - c_2\omega_2)\sin^2\theta}{c_2\alpha^2 + (2\omega_1^2\omega_2 - 2\alpha^2\omega_2 - c_2\omega_1^2)\sin^2\theta}, \tag{3.16}$$

and

$$\frac{\partial\theta}{\partial\xi^1} = \frac{\alpha\sqrt{(\omega_2^2-\omega_1^2)\{4\alpha^2-(c_2-2\omega_2)^2\}}\sin\theta\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_0}}{c_2\alpha^2+(2\omega_1^2\omega_2-2\alpha^2\omega_2-c_2\omega_1^2)\sin^2\theta}. \tag{3.17}$$

Where we have the upper limit on θ in the form,

$$\sin\theta_0 = \frac{c_2\alpha}{\sqrt{(\omega_2^2-\omega_1^2)\{4\alpha^2-(c_2-2\omega_2)^2\}}}. \tag{3.18}$$

Now we can explicitly solve for θ equation from (3.17) to get the profile of the string, which gives

$$\begin{aligned}\alpha\sqrt{(\omega_2^2-\omega_1^2)\{4\alpha^2-(c_2-2\omega_2)^2\}}\;\xi^1 &= \frac{c_2(\alpha^2-\omega_1^2)+(2\omega_1^2\omega_2-2\alpha^2\omega_2)}{\cos\theta_0}\cosh^{-1}\left[\frac{\cos\theta_0}{\cos\theta}\right]\\ &\quad-\frac{c_2\alpha^2}{\sin\theta_0}\cos^{-1}\left[\frac{\sin\theta_0}{\sin\theta}\right].\end{aligned} \tag{3.19}$$

From the above we can see $\theta=\theta_0$ can be thought of as the point exactly where the tip of the spike is situated. On the other hand it is clear that at $\theta\to\frac{\pi}{2}$ we actually have $\xi_1\to\infty$, making this a valid single spike solution. The largest spike goes up to the pole of the sphere where $\theta_0=0$. Now we can calculate the conserved charges for the D-string motion using usual noether techniques,

$$\begin{aligned}E &= -2\int_{\theta_0}^{\theta_1}\frac{\partial\mathcal{L}}{\partial_0 t}\frac{d\theta}{\partial_1\theta} = \frac{2T_1k\kappa(\alpha^2-\omega_1^2)(c_2-2\omega_2)}{\alpha^2\sqrt{(\omega_2^2-\omega_1^2)\{4\alpha^2-(c_2-2\omega_2)^2\}}}\int_{\theta_0}^{\frac{\pi}{2}}\frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_0}},\\ P_i &= 2\int_{\theta_0}^{\theta_1}\frac{\partial\mathcal{L}}{\partial_0 x_i}\frac{d\theta}{\partial_1\theta} = \frac{2T_1k\nu_i(\alpha^2-\omega_1^2)(c_2-2\omega_2)}{\alpha^2\sqrt{(\omega_2^2-\omega_1^2)\{4\alpha^2-(c_2-2\omega_2)^2\}}}\int_{\theta_0}^{\frac{\pi}{2}}\frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_0}},\\ D &= 2\int_{\theta_0}^{\theta_1}\frac{\partial\mathcal{L}}{\partial_0\rho}\frac{d\theta}{\partial_1\theta} = \frac{2T_1km(\alpha^2-\omega_1^2)(c_2-2\omega_2)}{\alpha^2\sqrt{(\omega_2^2-\omega_1^2)\{4\alpha^2-(c_2-2\omega_2)^2\}}}\int_{\theta_0}^{\frac{\pi}{2}}\frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_0}}.\end{aligned} \tag{3.20}$$

Here as usual, the E corresponds to time translation invariance, and P_i's correspond to shifts in x_i's. The other charge D comes from translations in $\rho=\ln r$. In general this is not a symmetry of the metric itself, but the way we have chosen the parameterisations in (3.10) makes it clear that these are indeed symmetries of the D1 string action. All these quantities diverge due to the divergent integral. The angular momenta J_1 and J_2 that arise from the isometries along ϕ_1 and ϕ_2 are given by,

$$\begin{aligned}J_1 = &\frac{2T_1k\omega_1[2(2\omega_2^2-2\alpha^2-c_2\omega_2)-c_2\alpha]}{\alpha\sqrt{(\omega_2^2-\omega_1^2)\{4\alpha^2-(c_2-2\omega_2)^2\}}}\int_{\theta_0}^{\frac{\pi}{2}}\frac{\sin\theta\cos\theta d\theta}{\sqrt{\sin^2\theta-\sin^2\theta_0}}\\ &-\frac{4T_1k\omega_1[(2\omega_2^2-2\alpha^2-c_2\omega_2)]}{\alpha\sqrt{(\omega_2^2-\omega_1^2)\{4\alpha^2-(c_2-2\omega_2)^2\}}}\int_{\theta_0}^{\frac{\pi}{2}}\frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_0}}.\end{aligned} \tag{3.21}$$

$$\begin{aligned}J_2 = &\frac{2T_1k[\alpha(2\omega_1^2-2\omega_2^2+c_2\omega_2)-2(2\omega_1^2\omega_2-2\alpha^2\omega_2-c_2\omega_1^2)]}{\alpha\sqrt{(\omega_2^2-\omega_1^2)\{4\alpha^2-(c_2-2\omega_2)^2\}}}\int_{\theta_0}^{\frac{\pi}{2}}\frac{\sin\theta\cos\theta d\theta}{\sqrt{\sin^2\theta-\sin^2\theta_0}}\\ &+\frac{2T_1k[2(c_2\alpha^2+2\omega_1^2\omega_2-2\alpha^2\omega_2-c_2\omega_1^2)]}{\alpha\sqrt{(\omega_2^2-\omega_1^2)\{4\alpha^2-(c_2-2\omega_2)^2\}}}\int_{\theta_0}^{\frac{\pi}{2}}\frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_0}}.\end{aligned} \tag{3.22}$$

The other quantity of interest is the angle deficit defined as $\Delta\phi = 2\int_{\theta_0}^{\theta_1} \frac{d\theta}{\partial_1\theta}$ is given by,

$$\Delta\phi = \frac{2}{\alpha\sqrt{(\omega_2^2-\omega_1^2)\{4\alpha^2-(c_2-2\omega_2)^2\}}}\Bigg[(c_2-2\omega_2)(\alpha^2-\omega_1^2)\int_{\theta_0}^{\frac{\pi}{2}}\frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_0}} + c_2\alpha^2\int_{\theta_0}^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta\sqrt{\sin^2\theta-\sin^2\theta_0}}\Bigg], \tag{3.23}$$

which is clearly divergent because of the first integral. However one can regularize the angle difference using a combination of other conserved charges to remove the divergent part,

$$(\Delta\phi)_{\rm reg} = \Delta\phi - \frac{1}{T_1 k}\sqrt{E^2 - D^2 - \sum_i P_i^2} = 2\cos^{-1}(\sin\theta_0)\,, \tag{3.24}$$

which implies

$$\sin\theta_0 = \cos\left(\frac{(\Delta\phi)_{\rm reg}}{2}\right). \tag{3.25}$$

We can also regularize both the angular momenta as,

$$\begin{aligned}(J_1)_{\rm reg} &= J_1 + \frac{2\omega_1(2\omega_2^2-2\alpha^2-c_2\omega_2)}{(c_2-2\omega_2)(\alpha^2-\omega_1^2)}\sqrt{E^2-\sum P_i^2 - D^2}\\ &= \frac{2T_1k\omega_1(4\omega_2^2-4\alpha^2-2c_2\omega_2-c_2\alpha)}{\alpha\sqrt{(\omega_2^2-\omega_1^2)\{4\alpha^2-(c_2-2\omega_2)^2\}}}\cos\theta_0\end{aligned} \tag{3.26}$$

and,

$$\begin{aligned}(J_2)_{\rm reg} &= J_2 - \frac{2(c_2\alpha^2+2\omega_1^2\omega_2-2\alpha^2\omega_2-c_2\omega_1^2)}{(c_2-2\omega_2)(\alpha^2-\omega_1^2)}\sqrt{E^2-\sum P_i^2-D^2}\\ &= \frac{2T_1k\{\alpha(2\omega_1^2-2\omega_2^2+c_2\omega_2)-2(2\omega_1^2\omega_2-2\alpha^2\omega_2-c_2\omega_1^2)\}}{\alpha\sqrt{(\omega_2^2-\omega_1^2)\{4\alpha^2-(c_2-2\omega_2)^2\}}}\cos\theta_0\end{aligned} \tag{3.27}$$

This regularisation for charges of a single-spike D1-string can be compared to similar construction of F-strings in a background B field, considered in for example, [44]. It can actually be shown that without the WZ term contribution, the angular momentum J_1 and J_2 are finite as can also be seen in [19]. But with finite background flux, they diverge and we have to regularise the charges to get a finite answer. These regularised angular momenta can be easily found to satisfy the dispersion relation,

$$(J_2)_{\rm reg} = \sqrt{(J_1)_{\rm reg}^2 + f_1(\lambda)\sin^2\left(\frac{(\Delta\phi)_{\rm reg}}{2}\right)}\,, \tag{3.28}$$

where $f_1(\lambda)$ is a complicated function of various constants and winding numbers and $N = \sqrt{\lambda}$ is the effective 't Hooft coupling. However, if we choose $c_2 = 2\omega_2 - \alpha$, then the complicated function reduces to $f_1(\lambda) = \frac{3\lambda}{\pi^2}$ and the dispersion relation looks like,

$$(J_2)_{\rm reg} = \sqrt{(J_1)_{\rm reg}^2 + \frac{3\lambda}{\pi^2}\sin^2\left(\frac{(\Delta\phi)_{\rm reg}}{2}\right)}\,, \tag{3.29}$$

which matches exactly with the dispersion relation obtained by studying fundamental rotating single-spike string solutions on NS5-branes [41].

3.2 Giant Magnon

In this case we demand the situation where both $\frac{\partial\theta}{\partial\xi^1}$ and $\frac{\partial\phi_2}{\partial\xi^1}$ diverge. Using these conditions yield the relations between the constants $\alpha = \omega_1$ and $c_3 = -2\omega_1$. In this case also the integration constant c_2 remains undetermined. As in the previous case we will put the value of c_2 by hand to reduce the dispersion relation. Using these conditions we write down the equations in the form,

$$\frac{\partial\phi_2}{\partial\xi^1} = \frac{(2\omega_2^2 - 2\omega_1^2 - c_2\omega_2)\sin^2\theta}{c_2\omega_1\cos^2\theta}, \tag{3.30}$$

and

$$\frac{\partial\theta}{\partial\xi^1} = \frac{\sin\theta\sqrt{\sin^2\theta - \sin^2\theta_1}}{\cos\theta\sin\theta_1}, \tag{3.31}$$

where the upper limit for θ is,

$$\sin\theta_1 = \frac{c_1c_2\omega_1}{\sqrt{(\omega_2^2 - \omega_1^2)(4c_1^2\omega_1^2 - c_2^2\kappa^2 - 4\kappa^2\omega_2^2 + 4c_2\kappa^2\omega_2)}}. \tag{3.32}$$

We can now easily integrate the θ equation to find the string profile as follows

$$\xi^1 = \cos^{-1}\left[\frac{\sin\theta_1}{\sin\theta}\right], \tag{3.33}$$

which gives a finite range of ξ^1 with $|\xi^1| \le \frac{\pi}{2} - \theta_1$ as expected.

In this case we can construct the conserved charges in the usual sense,

$$\begin{aligned}
E &= \frac{2T_1k(\kappa^2 - c_1^2)(c_2 - 2\omega_2)\sin\theta_1}{c_1c_2\omega_1}\int_{\frac{\pi}{2}}^{\theta_1}\frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta - \sin^2\theta_1}},\\
P_i &= \frac{2T_1k\nu_i(\kappa^2 - c_1^2)(c_2 - 2\omega_2)\sin\theta_1}{c_1c_2\kappa\omega_1}\int_{\frac{\pi}{2}}^{\theta_1}\frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta - \sin^2\theta_1}},\\
D &= \frac{2T_1km(\kappa^2 - c_1^2)(c_2 - 2\omega_2)\sin\theta_1}{c_1c_2\kappa\omega_1}\int_{\frac{\pi}{2}}^{\theta_1}\frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta - \sin^2\theta_1}}.
\end{aligned} \tag{3.34}$$

As in single-spike case before all these quantities diverge due to the divergent integral. Among the angular momenta we can write,

$$\begin{aligned}
J_1 =& \frac{2T_1k\{\omega_1(\kappa^2 - c_1^2)(c_2 - 2\omega_2) + 2c_1\kappa(2\omega_2^2 - 2\omega_1^2 - c_2\omega_2)\}}{c_1c_2\kappa\omega_1}\int_{\frac{\pi}{2}}^{\theta_1}\frac{\sin\theta_1\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta - \sin^2\theta_1}}\\
&- \frac{2T_1k\{2c_1\kappa(2\omega_2^2 - 2\omega_1^2 - c_2\omega_2) + \omega_1(2c_1^2\omega_2 + c_2\kappa^2 - 2\kappa^2\omega_2)\}}{c_1c_2\kappa\omega_1}\int_{\frac{\pi}{2}}^{\theta_1}\frac{\sin\theta_1\sin\theta\cos\theta d\theta}{\sqrt{\sin^2\theta - \sin^2\theta_1}},
\end{aligned} \tag{3.35}$$

Now this J_1 is divergent, but on the other hand,

$$J_2 = \frac{2T_1k[2c_1c_2\kappa\omega_1 - 2c_1^2\omega_1^2 - \kappa^2\omega_2(c_2 - 2\omega_2)]}{c_1c_2\kappa\omega_1}\sin\theta_1\cos\theta_1, \tag{3.36}$$

JHEP04(2016)172

is finite. Also the angle deficit can be shown to be finite,

$$\Delta\phi = -2\cos^{-1}(\sin\theta_1)\,. \tag{3.37}$$

which implies $\sin\theta_1 = \cos(\frac{(\Delta\phi)}{2})$. If we define the divergent quantity,

$$\tilde{E} = \frac{\omega_1(\kappa^2 - c_1^2)(c_2 - 2\omega_2) + 2c_1\kappa(2\omega_2^2 - 2\omega_1^2 - c_2\omega_2)}{\alpha(\kappa^2 - c_1^2)(c_2 - 2\omega_2)}\sqrt{E^2 - D^2 - \sum_i P_i^2}\,, \tag{3.38}$$

then the quantity,

$$\tilde{E} - J_1 = \frac{2T_1 k[2c_1\kappa(2\omega_1^2 - 2\omega_2^2 + c_2\omega_2) - \omega_1(2c_1^2\omega_2 + c_2\kappa^2 - 2\kappa^2\omega_2)]}{c_1 c_2 \kappa\omega_1}\sin\theta_1\cos\theta_1\,, \tag{3.39}$$

is finite. This exactly adheres to the case of the dyonic giant magnon i.e. bound state of J_2 number of giant magnons. In that case we also have E and J_1 divergent, but $E - J_1$ finite and J_2 held fixed. We can write the final dispersion relation in the form,

$$\tilde{E} - J_1 = \sqrt{J_2^2 + f_2(\lambda)\sin^2\left(\frac{\Delta\phi}{2}\right)}\,, \tag{3.40}$$

where again we have used the definition of effective 't Hooft coupling as before. Also $f_2(\lambda)$ remains very complicated as in the previous case. But if we choose $c_2\kappa = 2\kappa\omega_2 + c_1\omega_1$ then we can reduce it to $f_2(\lambda) = \frac{-3\lambda}{\pi^2}$ and the dispersion relation will become,

$$\tilde{E} - J_1 = \sqrt{J_2^2 - \frac{3\lambda}{\pi^2}\sin^2\left(\frac{\Delta\phi}{2}\right)}\,. \tag{3.41}$$

Once again one can see this relations exactly matches with the giant magnon dispersion relation obtained in [41].

4 Rotating D1-strings on intersecting D5-D5' branes

We shall now analyse the rotating D1-strings in the background of two stacks of D5-branes intersecting on a line. More precisely, we have N_1 D5-branes extended along $(x_0, x_1, x_2, x_3, x_4, x_5)$ direction and N_2 D5-branes extended along $(x_0, x_1, x_6, x_7, x_8, x_9)$ directions, having exactly eight relatively transverse dimensions. These N_1 and N_2 have to be large if we want the supergravity approximation to be valid, which constrain the regime of the coupling constant where the solution is valid. For this reason, most of the literature works with the S dual configuration [23]. Nonetheless, it is a very interesting solution having the background of the form,

$$ds^2 = (h_1 h_2)^{-\frac{1}{2}}(-dt^2 + dx_1^2) + h_1^{-\frac{1}{2}} h_2^{\frac{1}{2}} \sum_{i=2}^{5} dx_i^2 + h_1^{\frac{1}{2}} h_2^{-\frac{1}{2}} \sum_{i=6}^{9} dx_i^2\,, \tag{4.1}$$

together with a non-trivial dilaton,

$$e^{2\phi} = h_1^{-1} h_2^{-1}\,, \tag{4.2}$$

We have to remember that there are two sets of transverse spaces for the intersecting brane configuration. For the set of branes having worldvolume lying on $(x_0, x_1, x_2, x_3, x_4, x_5)$, the transverse space is along (x_6, x_7, x_8, x_9). We can measure the transverse radial direction in this case using $r_1^2 = \sum_{i=6}^{9} x_i^2$. Similarly for the second set of branes the transverse direction is parameterised by $r_2^2 = \sum_{i=2}^{5} x_i^2$. The harmonic functions for these set of branes are given by

$$h_1 = 1 + \frac{k_1}{r_1^2} \quad \text{and} \quad h_2 = 1 + \frac{k_2}{r_2^2} \tag{4.3}$$

with $k_1 = g_s N_1 l_s^2$, $k_2 = g_s N_2 l_s^2$ and also $\sum_{i=2}^{5} dx_i^2 = dr_2^2 + r_2^2 d\Omega_2^2$, $\sum_{i=6}^{9} dx_i^2 = dr_1^2 + r_1^2 d\Omega_1^2$. For this configuration it is natural to talk about dual 3-form RR fluxes which like the previous section can actually be constructed from the volume form of the transverse spheres (Ω_1, Ω_2) of the two sets of branes. Here we can parameterise the spheres as $d\Omega_1^2 = d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + \cos^2\theta_1 d\psi_1^2$ and $d\Omega_2^2 = d\theta_2^2 + \sin^2\theta_2 d\phi_2^2 + \cos^2\theta_2 d\psi_2^2$. Calculating the RR fluxes, we get

$$\tilde{C}_{\phi_1\psi_1} = 2k_1 \sin^2\theta_1\,, \qquad \tilde{C}_{\phi_2\psi_2} = 2k_2 \sin^2\theta_2\,. \tag{4.4}$$

Now in the near-horizon $(r_1 \to 0, r_2 \to 0)$, we can probe the region near the brane intersections. In this limit the harmonic functions approximate as $h_1 \approx \frac{k_1}{r_1^2}, h_2 \approx \frac{k_2}{r_2^2}$. The metric will reduce as,

$$ds^2 = \frac{r_1 r_2}{\sqrt{k_1 k_2}}\left[-dt^2 + dx_1^2 + k_1\left(\frac{dr_1^2}{r_1^2} + d\Omega_1^2\right) + k_2\left(\frac{dr_2^2}{r_2^2} + d\Omega_2^2\right)\right], \tag{4.5}$$

The dilaton in this limit becomes,

$$e^{-\phi} = \sqrt{h_1 h_2} = \frac{\sqrt{k_1 k_2}}{r_1 r_2}\,. \tag{4.6}$$

Before proceeding further we consider both the stacks contains same (large) number of D5-branes i.e., $N_1 = N_2 = N$, which implies $k_1 = k_2 = k$. With these simplification the metric, dilaton and magnetic dual RR fields become,

$$\begin{aligned} ds^2 &= \frac{r_1 r_2}{k}\left[-dt^2 + dx_1^2 + k\left(\frac{dr_1^2}{r_1^2} + d\Omega_1^2\right) + k\left(\frac{dr_2^2}{r_2^2} + d\Omega_2^2\right)\right], \\ e^{-\phi} &= \frac{k}{r_1 r_2}\,, \quad \tilde{C}_{\phi_1,\psi_1} = 2k\sin^2\theta_1\,, \quad \tilde{C}_{\phi_2,\psi_2} = 2k\sin^2\theta_2\,. \end{aligned} \tag{4.7}$$

Further, on rescaling $t \to \sqrt{k}t$ and $x_1 \to \sqrt{k}x_1$ and defining $\rho_1 = \ln r_1$ and $\rho_2 = \ln r_2$, the metric and the dilaton becomes,

$$\begin{aligned} ds^2 = e^{\rho_1} e^{\rho_2}\big[&-dt^2 + dx_1^2 + d\rho_1^2 + d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + \cos^2\theta_1 d\psi_1^2 \\ &+ d\rho_2^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2 + \cos^2\theta_2 d\psi_2^2\big]\,, \quad e^{-\phi} = k e^{-\rho_1} e^{-\rho_2}\,, \end{aligned} \tag{4.8}$$

while the magnetic dual RR field remains unchanged. Now, the Lagrangian-density can be written as,

$$
\begin{aligned}
\mathcal{L} &= -T_1 e^{-\phi}\sqrt{-\det\, A_{\alpha\beta}} + \frac{T_1}{2}\epsilon^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N C_{MN} \\
&= -T_1 k\Big[[-\partial_0 t\partial_1 t+\partial_0 x_1\partial_1 x_1+\partial_0\rho_1\partial_1\rho_1+\partial_0\theta_1\partial_1\theta_1+\sin^2\theta_1\partial_0\phi_1\partial_1\phi_1+\cos^2\theta_1\partial_0\psi_1\partial_1\psi_1 \\
&\quad + \partial_0\rho_2\partial_1\rho_2 + \partial_0\theta_2\partial_1\theta_2 + \sin^2\theta_2\partial_0\phi_2\partial_1\phi_2 + \cos^2\theta_2\partial_0\psi_2\partial_1\psi_2]^2 - [-(\partial_0 t)^2 + (\partial_0 x_1)^2 \\
&\quad + (\partial_0\rho_1)^2 + (\partial_0\theta_1)^2 + \sin^2\theta_1(\partial_0\phi_1)^2 + \cos^2\theta_1(\partial_0\psi_1)^2 + (\partial_0\rho_2)^2 + (\partial_0\theta_2)^2 \\
&\quad + \sin^2\theta_2(\partial_0\phi_2)^2+\cos^2\theta_2(\partial_0\psi_2)^2][-(\partial_1 t)^2+(\partial_1 x_2)^2+(\partial_1\rho_1)^2+(\partial_1\theta_1)^2+\sin^2\theta_1(\partial_1\phi_1)^2 \\
&\quad + \cos^2\theta_1(\partial_1\psi_1)^2 + (\partial_1\rho_2)^2 + (\partial_1\theta_2)^2 + \sin^2\theta_2(\partial_1\phi_2)^2 + \cos^2\theta_2(\partial_1\psi_2)^2]\Big]^{\frac{1}{2}} \\
&\quad + 2T_1 k\sin^2\theta_1(\partial_0\phi_1\partial_1\psi_1 - \partial_0\psi_1\partial_1\phi_1) + 2T_1 k\sin^2\theta_2(\partial_0\phi_2\partial_1\psi_2 - \partial_0\psi_2\partial_1\phi_2)\,. \qquad (4.9)
\end{aligned}
$$

Before solving the Euler-Lagrange equations, we assume the following rotating string ansatz,

$$
\begin{aligned}
&t = \kappa\xi^0\,, \qquad x_1 = v\xi^0\,, \quad \rho_i = m_i\xi^0\,, \quad \theta_i = \theta_i(\xi^1)\,, \\
&\phi_i = \nu_i\xi^0 + \xi^1\,, \quad \psi_i = \omega_i\xi^0 + \psi_i(\xi^1)\,, \qquad i = 1,2\,. \qquad (4.10)
\end{aligned}
$$

As we did in the last section also, these are basically generalizations of rotating F-string solutions which can be shown to be consistent with the equations of motion. Now by solving the equation of motion for t, we get,

$$
\frac{\kappa(\nu_1\sin^2\theta_1 + \omega_1\partial_1\psi_1\cos^2\theta_1 + \nu_2\sin^2\theta_2 + \omega_2\partial_1\psi_2\cos^2\theta_2)}{\sqrt{B}} = c_1\,, \qquad (4.11)
$$

where for convenience we have defined

$$
\begin{aligned}
B = {} & [\nu_1\sin^2\theta_1 + \omega_1\partial_1\psi_1\cos^2\theta_1 + \nu_2\sin^2\theta_2 + \omega_2\partial_1\psi_2\cos^2\theta_2]^2 - [-\alpha^2 + \nu_1^2\sin^2\theta_1 + \\
& \omega_1^2\cos^2\theta_1 + \nu_2^2\sin^2\theta_2 + \omega_2^2\cos^2\theta_2][(\partial_1\theta_1)^2 + (\partial_1\theta_2)^2 + \sin^2\theta_1 \\
& + \sin^2\theta_2 + \cos^2\theta_1(\partial_1\psi_1)^2 + \cos^2\theta_2(\partial_1\psi_2)^2] \qquad (4.12)
\end{aligned}
$$

and $\alpha^2 = \kappa^2 - v^2 - m_1^2 - m_2^2$ and c_1 are just constants.

Now solving the equations of motion for ϕ_1 and ψ_1, we get,

$$
\begin{aligned}
&\frac{\sin^2\theta_1}{\sqrt{B}}\Big[\omega_1(\omega_1 - \nu_1\partial_1\psi_1)\cos^2\theta_1 + \nu_2(\nu_2-\nu_1)\sin^2\theta_2 \\
&\qquad + \omega_2(\omega_2 - \nu_1\partial_1\psi_2)\cos^2\theta_2 - \alpha^2\Big] - 2\omega_1\sin^2\theta_1 = c_2\,, \qquad (4.13) \\
&\frac{\cos^2\theta_1}{\sqrt{B}}\Big[\nu_1(\nu_1\partial_1\psi_1 - \omega_1)\sin^2\theta_1 + \nu_2(\nu_2\partial_1\psi_1 - \omega_1)\sin^2\theta_2 \\
&\qquad + \omega_2(\omega_2\partial_1\psi_1 - \omega_1\partial_1\psi_2)\cos^2\theta_2 - \alpha^2\partial_1\psi_1\Big] + 2\nu_1\sin^2\theta_1 = c_3\,. \qquad (4.14)
\end{aligned}
$$

Again, solving for ϕ_2 and ψ_2, we get,

$$\frac{\sin^2\theta_2}{\sqrt{B}}\Big[\nu_1(\nu_1-\nu_2)\sin^2\theta_1+\omega_1(\omega_1-\nu_2\partial_1\psi_1)\cos^2\theta_1 \\ +\omega_2(\omega_2-\nu_2\partial_1\psi_2)\cos^2\theta_2-\alpha^2\Big]-2\omega_2\sin^2\theta_2=c_4\,, \tag{4.15}$$

$$\frac{\cos^2\theta_2}{\sqrt{B}}\Big[\nu_1(\nu_1\partial_1\psi_2-\omega_2)\sin^2\theta_1+\omega_1(\omega_1\partial_1\psi_2-\omega_2\partial_1\psi_1)\cos^2\theta_1 \\ +\nu_2(\nu_2\partial_1\psi_2-\omega_2)\sin^2\theta_2-\alpha^2\partial_1\psi_2\Big]+2\nu_2\sin^2\theta_2=c_5\,. \tag{4.16}$$

Solving (4.14) we get,

$$[c_1\nu_1^2\sin^2\theta_1+c_1\nu_2^2\sin^2\theta_2+c_1\omega_2^2\cos^2\theta_2+2\kappa\nu_1\omega_1\sin^2\theta_1-c_1\alpha^2-c_3\kappa\omega_1]\partial_1\psi_1\cos^2\theta_1 \\ =(c_1\omega_1\cos^2\theta_1-2\kappa\nu_1\sin^2\theta_1+c_3\kappa)(\omega_2\partial_1\psi_2\cos^2\theta_2+\nu_1\sin^2\theta_1+\nu_2\sin^2\theta_2)\,. \tag{4.17}$$

Again, solving (4.16) we get,

$$[c_1\nu_1^2\sin^2\theta_1+c_1\nu_2^2\sin^2\theta_2+c_1\omega_1^2\cos^2\theta_1+2\kappa\nu_2\omega_2\sin^2\theta_2-c_1\alpha^2-c_5\kappa\omega_2]\partial_1\psi_2\cos^2\theta_2 \\ =(c_1\omega_2\cos^2\theta_2-2\kappa\nu_2\sin^2\theta_2+c_5\kappa)(\omega_1\partial_1\psi_1\cos^2\theta_1+\nu_1\sin^2\theta_1+\nu_2\sin^2\theta_2)\,. \tag{4.18}$$

And finally from equation (4.11), we have,

$$(\partial_1\theta_1)^2+(\partial_1\theta_1)^2=\frac{(c_1^2-\kappa^2)(\nu_1\sin^2\theta_1+\omega_1\partial_1\psi_1\cos^2\theta_1+\nu_2\sin^2\theta_2+\omega_2\partial_1\psi_2\cos^2\theta_2)^2}{c_1^2(-\alpha^2+\nu_1^2\sin^2\theta_1+\omega_1^2\cos^2\theta_1+\nu_2^2\sin^2\theta_2+\omega_2^2\cos^2\theta_2)} \\ -\sin^2\theta_1-\sin^2\theta_2-\cos^2\theta_1(\partial_1\psi_1)^2-\cos^2\theta_2(\partial_1\psi_2)^2\,. \tag{4.19}$$

Equation (4.19) have two variables $\partial_1\theta_1$ and $\partial_1\theta_2$, it is quite difficult to find the solutions of both $\partial_1\theta_1$ and $\partial_1\theta_2$ simultaneously from the form of the equation (4.19). For simplicity we will now consider $\theta_1=\theta_2=\theta$, which will confine the string on both the spheres but with restriction that both the sphere will have the same value of θ. This will make our equations more tractable.

Also, we define the following quantities for our convenience,

$$\begin{aligned} a&=c_1(\nu_1^2+\nu_2^2)\sin^2\theta-c_1\alpha^2\,, & b&=c_1\omega_2^2\cos^2\theta+2\kappa\nu_1\omega_1\sin^2\theta-c_3\kappa\omega_1\,,\\ c&=c_1\omega_1^2\cos^2\theta+2\kappa\nu_2\omega_2\sin^2\theta-c_5\kappa\omega_2\,, & d&=(\nu_1+\nu_2)\sin^2\theta\,,\\ e&=c_1\omega_1\cos^2\theta-2\kappa\nu_1\sin^2\theta+c_3\kappa\,, & f&=c_1\omega_2\cos^2\theta-2\kappa\nu_2\sin^2\theta+c_5\kappa\,, \end{aligned} \tag{4.20}$$

then equation (4.17) and (4.18) can be written as,

$$\begin{aligned}(a+b)\partial_1\psi_1\cos^2\theta&=e(\omega_2\partial_1\psi_2\cos^2\theta+d)\,,\\ (a+c)\partial_1\psi_2\cos^2\theta&=f(\omega_1\partial_1\psi_1\cos^2\theta+d)\,.\end{aligned} \tag{4.21}$$

By solving the above equations we get,

$$\begin{aligned}\partial_1\psi_1&=\frac{ed(a+c+\omega_2 f)}{\cos^2\theta[(a+b)(a+c)-\omega_1\omega_2 ef]}\,,\\ \partial_1\psi_2&=\frac{fd(a+b+\omega_1 e)}{\cos^2\theta[(a+b)(a+c)-\omega_1\omega_2 ef]}\,.\end{aligned} \tag{4.22}$$

Again using these results equation (4.19) reduces to,

$$(\partial_1\theta)^2 = \frac{(\kappa^2-c_1^2)}{2c_1^2}\frac{[(\nu_1+\nu_2)\sin^2\theta+(\omega_1\partial_1\psi_1+\omega_2\partial_2\psi_2)\cos^2\theta]^2}{[\alpha^2-(\nu_1^2+\nu_2^2)\sin^2\theta-(\omega_1^2+\omega_2^2)\cos^2\theta]}$$
$$-\sin^2\theta-\frac{1}{2}[(\partial_1\psi_1)^2+(\partial_1\psi_2)^2]\cos^2\theta\,. \tag{4.23}$$

The above set of equations of motion appear to be generalizations of the ones discussed in the last section for single stack of D5 branes. We will exactly follow the way we found out the single spike and giant magnon solutions for the last section. Namely we will impose the similar kind of conditions on $\partial_1\theta$, $\partial_1\psi_1$ and $\partial_1\psi_2$ and define the conserved charges in each case separately.

4.1 Single Spike-like solution

For a generalised spike solution, we impose the condition on the derivatives

$$\partial_1\theta\to 0 \quad \text{as} \quad \theta\to\frac{\pi}{2}, \tag{4.24}$$

Implementing the conditions, from equation (4.23), we get,

$$c_1=\frac{\kappa(\nu_1+\nu_2)}{\alpha_1}\,, \quad \text{where} \quad \alpha_1=\sqrt{2\alpha^2-(\nu_1-\nu_2)^2}\,. \tag{4.25}$$

Again, follwing our earlier discussion we will demand that the derivatives $\partial_1\psi_1$ and $\partial_2\psi_2$ have to be regular in the limit $\theta\to\frac{\pi}{2}$. This leads us to the relations between the constants $c_3=2\nu_1$ and $c_5=2\nu_2$ respectively. Using these relations, we get,

$$\partial_1\psi_1=\frac{(\nu_1+\nu_2)(c_1\omega_1+2\kappa\nu_1)\sin^2\theta}{c_1(\nu_1^2+\nu_2^2)\sin^2\theta-2\kappa\alpha_2\cos^2\theta-c_1\alpha^2}\,,$$
$$\partial_1\psi_2=\frac{(\nu_1+\nu_2)(c_1\omega_2+2\kappa\nu_2)\sin^2\theta}{c_1(\nu_1^2+\nu_2^2)\sin^2\theta-2\kappa\alpha_2\cos^2\theta-c_1\alpha^2}\,, \tag{4.26}$$

where $\alpha_2=\nu_1\omega_1+\nu_2\omega_2$. Similarly, equation (4.23) reduces to,

$$\partial_1\theta=\sqrt{\frac{a_1}{2}}\frac{\kappa\sin\theta\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_0}}{\alpha_1[c_1(\nu_1^2+\nu_2^2)\sin^2\theta-2\kappa\alpha_2\cos^2\theta-c_1\alpha^2]}\,, \tag{4.27}$$

where

$$\sin\theta_0=\frac{2\alpha_1\alpha_2+(\nu_1+\nu_2)\alpha^2}{\sqrt{\frac{a_1}{2}}} \tag{4.28}$$

and

$$a_1=2\alpha^2(\nu_1^2+\nu_2^2)(\nu_1+\nu_2)^2+8\alpha_1^2\alpha_2^2+8\alpha_1\alpha_2(\nu_1+\nu_2)(\nu_1^2+\nu_2^2)-4\alpha_1\alpha_2(\nu_1+\nu_2)^3$$
$$-\alpha_1^2(\nu_1+\nu_2)(\omega_1^2+\omega_2^2+4\nu_1^2+\nu_2^2). \tag{4.29}$$

Now, like the previous section we can solve for the string profile by integrating the θ equation. We will not present the cumbersome expressions here for brevity.

JHEP04(2016)172

Let us now define the conserved charges corresponding to the various isometries as given by,

$$
\begin{aligned}
E &= -2T_1 \int \frac{\partial \mathcal{L}}{\partial(\partial_0 t)} \frac{d\theta}{\partial_1 \theta} = \frac{4T_1 k\kappa(\nu_1+\nu_2)(\nu_1^2+\nu_2^2-\alpha^2)}{\alpha_1\sqrt{\frac{a_1}{2}}} \int_{\theta_0}^{\frac{\pi}{2}} \frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_0}}\,, \\
P &= 2T_1 \int \frac{\partial \mathcal{L}}{\partial(\partial_0 x_1)} \frac{d\theta}{\partial_1 \theta} = \frac{4T_1 kv(\nu_1+\nu_2)(\nu_1^2+\nu_2^2-\alpha^2)}{\alpha_1\sqrt{\frac{a_1}{2}}} \int_{\theta_0}^{\frac{\pi}{2}} \frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_0}}\,, \\
D_1 &= 2T_1 \int \frac{\partial \mathcal{L}}{\partial(\partial_0 \rho_1)} \frac{d\theta}{\partial_1 \theta} = \frac{4T_1 km_1(\nu_1+\nu_2)(\nu_1^2+\nu_2^2-\alpha^2)}{\alpha_1\sqrt{\frac{a_1}{2}}} \int_{\theta_0}^{\frac{\pi}{2}} \frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_0}}\,, \\
D_2 &= 2T_1 \int \frac{\partial \mathcal{L}}{\partial(\partial_0 \rho_2)} \frac{d\theta}{\partial_1 \theta} = \frac{4T_1 km_2(\nu_1+\nu_2)(\nu_1^2+\nu_2^2-\alpha^2)}{\alpha_1\sqrt{\frac{a_1}{2}}} \int_{\theta_0}^{\frac{\pi}{2}} \frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_0}}\,.
\end{aligned}
\tag{4.30}
$$

Like the previous section, the charges D_1 and D_2 are associated with shifts in $\ln r_1$ and $\ln r_2$, which again are symmetries of the D1 string action. All these above quantities can be shown to diverge. But, using these expressions we can define a new divergent quantity,

$$
\sqrt{E^2-P^2-D_1^2-D_2^2} = \frac{4T_1 k\alpha(\nu_1+\nu_2)(\nu_1^2+\nu_2^2-\alpha^2)}{\alpha_1\sqrt{\frac{a_1}{2}}} \int_{\theta_0}^{\frac{\pi}{2}} \frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_0}}\,.
\tag{4.31}
$$

The angle deficit $\Delta\phi = 2\int \frac{d\theta}{\partial_1\theta}$ is given by,

$$
\begin{aligned}
\Delta\phi = \frac{2\alpha_1}{\kappa\sqrt{\frac{h_1}{2}}} \Bigg[& c_1(\nu_1^2+\nu_2^2-\alpha^2) \int_{\theta_0}^{\frac{\pi}{2}} \frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_0}} \\
& -(c_1\alpha^2+2\kappa\alpha_2) \int_{\theta_0}^{\frac{\pi}{2}} \frac{\sin\theta\cos\theta d\theta}{\sqrt{\sin^2\theta-\sin^2\theta_0}} \Bigg]\,.
\end{aligned}
\tag{4.32}
$$

This is also divergent, but we can regularize it by removing the divergent part,

$$
(\Delta\phi)_{\text{reg}} = \Delta\phi - \frac{\alpha_1}{2T_1 k\alpha}\sqrt{E^2-P^2-D_1^2-D_2^2} = -\cos^{-1}(\sin\theta_0)\,,
\tag{4.33}
$$

which implies $\sin\theta_0 = \cos(\frac{(\Delta\phi)_{\text{reg}}}{2})$. Angular momenta $J_1 = 2T_1\int \frac{\partial\mathcal{L}}{\partial(\partial_1\phi_1)}\frac{d\theta}{\partial_1\theta}$ and $J_2 = 2T_1\int \frac{\partial\mathcal{L}}{\partial(\partial_1\phi_2)}\frac{d\theta}{\partial_1\theta}$ also diverge. Regularised values of J_1 and J_2 are given by,

$$
\begin{aligned}
(J_1)_{\text{reg}} &= J_1 - \frac{2\alpha_1(\nu_1+\nu_2)\omega_1+4\alpha_1^2\nu_1+(\nu_1-\nu_2)(\nu_1^2+\nu_2^2-\alpha^2)}{2\alpha(\nu_1^2+\nu_2^2-\alpha^2)}\sqrt{E^2-P^2-D_1^2-D_2^2} \\
&= \frac{2T_1 k(\nu_1+\nu_2)}{\alpha_1\sqrt{\frac{a_1}{2}}}[2\alpha_1\nu_2(\omega_2-\omega_1)+2\nu_1(\alpha^2-2\alpha_1^2)-(\nu_1-\nu_2)(\nu_1^2+\nu_2^2)]\cos\theta_0\,, \\
(J_2)_{\text{reg}} &= J_2 - \frac{2\alpha_1(\nu_1+\nu_2)\omega_2+4\alpha_1^2\nu_1+(\nu_2-\nu_1)(\nu_1^2+\nu_2^2-\alpha^2)}{2\alpha(\nu_1^2+\nu_2^2-\alpha^2)}\sqrt{E^2-P^2-D_1^2-D_2^2} \\
&= \frac{2T_1 k(\nu_1+\nu_2)}{\alpha_1\sqrt{\frac{a_1}{2}}}[2\alpha_1\nu_1(\omega_1-\omega_2)+2\nu_2(\alpha^2-2\alpha_1^2)+(\nu_1-\nu_2)(\nu_1^2+\nu_2^2)]\cos\theta_0\,.
\end{aligned}
\tag{4.34}
$$

JHEP04(2016)172

Again the angular momenta $K_1 = 2T_1 \int \frac{\partial \mathcal{L}}{\partial(\partial_1 \psi_1)} \frac{d\theta}{\partial_1 \theta}$ and $K_2 = 2T_1 \int \frac{\partial \mathcal{L}}{\partial(\partial_1 \psi_2)} \frac{d\theta}{\partial_1 \theta}$ also diverges. Regularised K_1 and K_2 are given by,

$$\begin{aligned}
(K_1)_{\text{reg}} &= K_1 + \frac{\alpha_1}{\alpha}\sqrt{E^2 - P^2 - D_1^2 - D_2^2} \\
&= \frac{2T_1 k}{\sqrt{\frac{a_1}{2}}}[4\alpha_1\alpha_2 - (\nu_1+\nu_2)\omega_1\alpha_1 - 2\nu_2^2(\nu_1^2 - \nu_2^2)]\cos\theta_0\,, \\
(K_2)_{\text{reg}} &= K_2 - \frac{\alpha_1}{\alpha}\sqrt{E^2 - P^2 - D_1^2 - D_2^2} \\
&= \frac{2T_1 k}{\sqrt{\frac{a_1}{2}}}[4\alpha_1\alpha_2 - (\nu_1+\nu_2)\omega_2\alpha_1 - 2\nu_1^2(\nu_2^2 - \nu_1^2)]\cos\theta_0\,,
\end{aligned} \tag{4.35}$$

Now, defining $J_{\text{reg}} = (J_1)_{\text{reg}} + (J_2)_{\text{reg}}$ and $K_{\text{reg}} = (K_1)_{\text{reg}} - (K_2)_{\text{reg}}$, we find that they satisfy a generalized dispersion relation of form,

$$J_{\text{reg}} = \sqrt{K_{\text{reg}}^2 + f_3(\lambda)\sin^2\left(\frac{(\Delta\phi)_{\text{reg}}}{2}\right)}\,, \tag{4.36}$$

where $f_3(\lambda) = \frac{2\lambda}{\pi^2}\frac{(\nu_1+\nu_2)^2}{a_1\alpha_1^2}[\{2\alpha_1(\omega_2-\omega_1)(\nu_2-\nu_1)+2(\nu_1+\nu_2)(\alpha^2-2\alpha_1^2)\}^2 - \alpha_1^2\{(\omega_2-\omega_1)\alpha_1 + (\nu_2^2-\nu_1^2)\}^2]$.

4.2 Giant Magnon-like solution

Here we use the opposite condition on the equations of motion, i.e. we impose that $\partial_1\theta$, $\partial_1\psi_1$ and $\partial_1\psi_2$ diverges as $\theta \to \frac{\pi}{2}$. Imposing the first one, we get the following condition,

$$\alpha^2 = \nu_1^2 + \nu_2^2 \tag{4.37}$$

we can not determine the values of c_3 and c_5 by this condition alone. As we have found in the previous cases, we put $c_3 = 2\nu_1$ and $c_5 = 2\nu_2$ by hand. Using these values of constants, we get,

$$\begin{aligned}
\partial_1\psi_1 &= -\frac{(\nu_1+\nu_2)(c_1\omega_1 + 2\kappa\nu_1)\sin^2\theta}{\beta\cos^2\theta}\,, \\
\partial_1\psi_2 &= -\frac{(\nu_1+\nu_2)(c_1\omega_2 + 2\kappa\nu_2)\sin^2\theta}{\beta\cos^2\theta}\,,
\end{aligned} \tag{4.38}$$

where $\beta = c_1(\nu_1^2+\nu_2^2) + 2\kappa\alpha_2$. Also, equation (4.23) reduces to,

$$\partial_1\theta = \frac{\sin\theta\sqrt{\sin^2\theta - \sin^2\theta_1}}{\sin\theta_1\cos\theta}\,, \tag{4.39}$$

where $\sin\theta_1 = \frac{\beta}{\sqrt{\frac{a_2}{2}}}$ and $a_2 = 2\beta^2 - (\nu_1+\nu_2)^2[(3\kappa^2 + c_1^2)(\nu_1^2+\nu_2^2) + \kappa^2(\omega_1^2+\omega_2^2+4c_2\kappa\alpha_2)]$.

In this case, the conserved charges are given by,

$$
\begin{aligned}
E &= \frac{2T_1 k(\nu_1+\nu_2)(c_1^2-\kappa^2)}{\sqrt{\frac{a_2}{2}}}\int_{\frac{\pi}{2}}^{\theta_1}\frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_1}}\,,\\
P &= \frac{2T_1 kv(\nu_1+\nu_2)(c_1^2-\kappa^2)}{\kappa\sqrt{\frac{a_2}{2}}}\int_{\frac{\pi}{2}}^{\theta_1}\frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_1}}\,,\\
D_1 &= \frac{2T_1 km_1(\nu_1+\nu_2)(c_1^2-\kappa^2)}{\kappa\sqrt{\frac{a_2}{2}}}\int_{\frac{\pi}{2}}^{\theta_1}\frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_1}}\,,\\
D_2 &= \frac{2T_1 km_2(\nu_1+\nu_2)(c_1^2-\kappa^2)}{\kappa\sqrt{\frac{a_2}{2}}}\int_{\frac{\pi}{2}}^{\theta_1}\frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_1}}\,.
\end{aligned}
\tag{4.40}
$$

All these quantities diverge and we again define the quantity,

$$
\sqrt{E^2-P^2-D_1^2-D_2^2}=\frac{2T_1 k\alpha(\nu_1+\nu_2)(c_1^2-\kappa^2)}{\kappa\sqrt{\frac{a_2}{2}}}\int_{\frac{\pi}{2}}^{\theta_1}\frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_1}}\,. \tag{4.41}
$$

The angle deficit is finite and is given by,

$$
\Delta\phi = 2\sin\theta_1\int_{\frac{\pi}{2}}^{\theta_1}\frac{\cos\theta d\theta}{\sin\theta\sqrt{\sin^2\theta-\sin^2\theta_1}} = 2\cos^{-1}(\sin\theta_1)\,, \tag{4.42}
$$

which implies $\sin\theta_1=\cos(\frac{\Delta\phi}{2})$. Again, the angular momenta J_1 and J_2 are given by,

$$
\begin{aligned}
J_1 = \frac{2T_1 k}{\kappa\sqrt{\frac{a_2}{2}}}\Bigg[&[c_1\beta+(\nu_1+\nu_2)\{2\kappa(c_1\omega_1+2\kappa\nu_1)-\nu_1(c_1^2-\kappa^2)\}]\int_{\frac{\pi}{2}}^{\theta_1}\frac{\sin\theta\cos\theta d\theta}{\sqrt{\sin^2\theta-\sin^2\theta_1}}\\
&+(\nu_1+\nu_2)[\nu_1(c_1^2-\kappa^2)-2\kappa(c_1\omega_1+2\kappa\nu_1)]\int_{\frac{\pi}{2}}^{\theta_1}\frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_1}}\Bigg]\,,
\end{aligned}
\tag{4.43}
$$

$$
\begin{aligned}
J_2 = \frac{2T_1 k}{\kappa\sqrt{\frac{a_2}{2}}}\Bigg[&[c_1\beta+(\nu_1+\nu_2)\{2\kappa(c_1\omega_2+2\kappa\nu_2)-\nu_2(c_1^2-\kappa^2)\}]\int_{\frac{\pi}{2}}^{\theta_1}\frac{\sin\theta\cos\theta d\theta}{\sqrt{\sin^2\theta-\sin^2\theta_1}}\\
&+(\nu_1+\nu_2)[\nu_2(c_1^2-\kappa^2)-2\kappa(c_1\omega_2+2\kappa\nu_2)]\int_{\frac{\pi}{2}}^{\theta_1}\frac{\sin\theta d\theta}{\cos\theta\sqrt{\sin^2\theta-\sin^2\theta_1}}\Bigg]\,,
\end{aligned}
\tag{4.44}
$$

But, the angular momenta K_1 and K_2 are finite and are given by,

$$
\begin{aligned}
K_1 &= \frac{2T_1 k}{\sqrt{\frac{a_2}{2}}}[(\nu_1+\nu_2)(\omega_1\kappa+2c_1\nu_1)+2\beta]\cos\theta_1\,,\\
K_2 &= \frac{2T_1 k}{\sqrt{\frac{a_2}{2}}}[(\nu_1+\nu_2)(\omega_2\kappa+2c_1\nu_2)+2\beta]\cos\theta_1\,,
\end{aligned}
\tag{4.45}
$$

Now, defining $J=J_1+J_2$, $K=K_1-K_2$, and

$$
\tilde{E}=\frac{(\nu_1-\nu_2)(c_1^2-5\kappa^2)+2c_1\kappa(\omega_1-\omega_2)}{\alpha(c_1^2-\kappa^2)}\sqrt{E^2-P^2-D_1^2-D_2^2}\,, \tag{4.46}
$$

we find that they satisfy a dispersion relation of form,

$$
\tilde{E}-J=\sqrt{K^2+f_4(\lambda)\sin^2\left(\frac{\Delta\phi}{2}\right)}\,, \tag{4.47}
$$

where $f_4(\lambda) = \frac{2\lambda}{\pi^2}\frac{(\nu_1+\nu_2)^2}{a_2\kappa^2}[\{(\nu_2-\nu_1)(c_1^2-5\kappa^2)+2c_1\kappa(\omega_1-\omega_2)\}^2-\kappa^2\{\kappa(\omega_1-\omega_2)+2c_1(\nu_1+\nu_2)\}^2]$. Again, it is noteworthy that the same form of generalised giant magnon solution was found for a rotating F-string in [42].

5 Conclusions

In this paper, we have studied various solutions of the D-string equations of motion in various curved backgrounds. First we have studied string equations of motion, of a bound state of oscillating D1 strings and F-strings with non-trivial gauge field on the D1 world-volume, in the recently found AdS_3 background with mixed fluxes. One of the interesting outcome of this solution is that the periodically expanding and contracting $(1, n)$ string has a possibility of reaching the boundary of AdS_3 in finite time in contrast to the probe D-string motion in the WZW model with only NS-NS fluxes [16]. Further we have studied the DBI equations of motion of the D1-string in D5-brane background and found out two classes of solutions corresponding to the giant magnon and single spike solutions of the D-string. Also, we have studied the rigidly rotating D1-string solution in the intersecting D5-brane background and compared our results with the existing S-dual configurations. These solutions are basically generalisations of the usual single spike and giant magnon solutions for a D1-string. However, the physical interpretation of such string states remain elusive from us as of now.

Acknowledgments

It is a pleasure to thank Kamal L. Panigrahi for discussions and important suggestions on the manuscript. The authors are indebted to S. Pratik Khastgir for various fruitful discussions. AB would like to thank Pulastya Parekh and Abhishake Sadhukhan for their comments. He would also like to thank the organisers of National String Meeting 2015 for kind hospitality at IISER Mohali, where a part of this work was done.

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JHEP04(2016)172